ON THE ASYMPTOTIC INTEGRATION OF A SYSTEM OF NONLINEAR EQUATIONS OF PLATE THEORY

(OB ASIMPTOTICHESKOM INTEGRIROVANII SISTEMY NELINEINYKH URAVNENII TEORII PLASTIN)

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L.S. SRUBSHCHIK
(Rostov-on-Don)

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The system of equations of the nonlinear plate theory contains an intrinsic small parameter ε^2 (the relative plate thinness). In the case when only the tensile stresses are developed in the corresponding membrane ($\varepsilon = 0$), it has been shown by asymptotic methods that for small ε there exists a state of equilibrium in which the plate behaves like a membrane everywhere except in the narrow region near the boundary where the edge effect takes place. A method of obtaining this solution has been constructed at the same time. The general results obtained are then particularized for the cases of an axisymmetrical plate and a plate of arbitrary shape subjected to tensile stresses on the contour.

1. On the formulation of the problem. We consider a system of nonlinear differential equations of the theory of flexible plates [1] due to Kármán

$$\Delta^2 F + {}^1/_2\,[w, w] = 0, \qquad \varepsilon^2 \Delta^2 w - [w, F] - q = 0 \qquad (1.1)$$

$$[w, F] \equiv w_{xx} F_{yy} + w_{yy} F_{xx} - 2 w_{xy} F_{xy} \qquad (1.2)$$

$$\left(w = \frac{W}{\sqrt{E}}, \quad \varepsilon^2 = \frac{Eh^2}{12(1-\nu^2)}, \quad q = q_1 \frac{\sqrt{E}}{h} \quad 0 < \nu < 0.5 \right) \qquad (1.3)$$

Here F is the stress function, W is the deflection of points of the middle surface. The quantity ε^2 characterizes the relative plate thinness, h is the plate thickness, E is Young's modulus and ν is Poisson's ratio, $q_1(x,y)$ is the magnitude of the external load which is acting along the normal to the plate surface. It is assumed that $q_1(x,y)$ is a sufficiently smooth function.

Let a plate occupy a bounded region D with a sufficiently smooth contour Γ. Moreover, it is assumed that on the contour*

* The case of a rigidly built-in plate has been chosen merely for the sake of definiteness: what follows can be easily applied to some other common cases, e.g. hinge supports.

$$w|_\Gamma = 0, \qquad w_n|_\Gamma = 0 \tag{1.4}$$

$$F_{\tau\tau}|_\Gamma = T(A) \geqslant 0, \qquad F_{n\tau}|_\Gamma = S(A) \qquad (A \in \Gamma) \tag{1.5}$$

Here n and τ are the normal and tangential directions on the bounary, and $F_{\tau\tau}(A)$ and $F_{n\tau}(A)$ are, respectively, the normal and tangential components of the external traction applied to the plate contour. It is assumed that the system of forces applied to the contour of the plate satisfies the conditions of equilibrium and compatibility. Then the existence of the solution of the problem (1.1) to (1.5) follows from the results of [2 and 3].

Along with the problem (1.1) to (1.5) we will consider a "degenerate" problem (on the equilibrium of a membrane)

$$\Delta^2 F_0 + {}^1/_2\,[w_0, w_0] = 0, \qquad -[w_0, F_0] - q = 0 \tag{1.6}$$

$$w_0|_\Gamma = 0, \qquad F_{0,\tau\tau}|_\Gamma = T(A), \qquad F_{0,n\tau}|_\Gamma = S(A) \qquad (A \in \Gamma) \tag{1.7}$$

Here as well as everywhere henceforth the indices following a comma designate differentiation with respect to corresponding variables.

Consider the problem of the asymtotic behavior of the solutions of (1.1) to (1.5) when $\epsilon \to 0$. In the case of a circular-symmetrically loaded plate with various support conditions, asymptotic representations were constructed for a symmetric solution in [4 to 7], and it was established that the solution is close to the solution of the degenerate problem ($\epsilon = 0$) everywhere, except in the narrow vicinity of the boundary, where the edge effect takes place. In [8] Fife has investigated the asymptotic expansion of the solution of (1.1) to (1.5) for the case of a rigidly built-in plate of arbitrary shape, subjected to uniform normal extension on the contour

$$F_{\tau\tau}|_\Gamma = \sigma \equiv \text{const}, \qquad F_{n\tau}|_\Gamma = 0 \tag{1.8}$$

In the same way as in the case of radial symmetry, it was established that for $\epsilon \to 0$ the solution of the problem (1.1) to (1.4) and (1.8) converges uniformly to the solution of the "degenerate" problem everywhere except in the close vicinity of the boundary. The quantity $p = q\sigma^{-3/2}$ is here assumed to be sufficiently small, which permits the use of the method of successive approximations, together with the existence of solutions of the two problems, to prove their uniqueness. On the other hand, it is well known [9 to 11] that a loaded plate or a shell has, generally speaking, more than one form of equilibrium.

Hence, the question naturally arises which of the solutions of problem (1.1) to (1.5) are "close", in the above sense, to the solutions of problem (1.6) and (1.7), when $\epsilon \to 0$.

In the membrane only the tensile stresses are developed *. Therefore,

* In the literature such membranes are referred to as nonmetallic (see [20 and 21]).

the solutions of problem (1.6) and (1.7) which are mechanically meaningful are those which at every point of the region D satisfy conditions

$$F_{0,xx} = \sigma_y > 0, \quad F_{0,yy} = \sigma_x > 0, \quad F_{0,xy} = -\tau, \ \sigma_x\sigma_y - \tau^2 > 0 \tag{1.9}$$

In what follows such solutions are referred to as positive. It is natural to look for solutions of problem (1.1) to (1.5) which are close to the positive solutions of problem (1.6) and (1.7). We will consider solutions of problem (1.1) to (1.5) for which a condition of the form (1.9) is satisfied, and will refer to them as membrane solutions.

It has been shown in the present paper, that if the positive solution of problem (1.6) and (1.7) exists, then the membrane solution for the plate also exists. It is unique (Theorem 3.2); asymptotic expansions are constructed for it, and estimates of errors are established for $\varepsilon \to 0$ (Theorem 3.3). Namely, when $\varepsilon \to 0$ the membrane solutions become positive solutions everywhere, except in the close vicinity of the boundary where the edge effect takes place. The proof of the above facts is furnished by constructing the asymptotic expansions of the solution of problem (1.1) to (1.5), analogous to those obtained in [6 and 7] for the case of radial symmetry (Section 2), and by applying the Newton's method which was developed for the operator equations by Kantorovich [12].

It becomes obvious from what has been said above, that the existence of the positive solution for a membrane is an important factor. Such existence can be established in a number of cases. For instance, it exists (a) in the case of a symmetric plate (Theorems 4.1 and 4.2) and shells of revolution [13], (b) in the case (1.6) and (1.8) and in analogous more general ones (Theorems 4.3 to 4.5). In the general case of problem (1.6) and (1.7) there are no positive solutions. These questions are discussed in Section 4.

Let us point out that, unlike in [8], here the argumentation of the asymptotic expansions is not related to the uniqueness of solution (corollaries of Theorem 4.1). Moreover, in Section 2 we consider a case in which the edge effect is described by boundary layers of a fractional order. This is the case of a rigidly built-in plate whose contour is free of stresses. In the general case, when problem (1.6) and (1.7) does not have the positive solution, the degeneration is of a more complicated character. One of the examples of this nature is considered by Fredrichs and Stoker [14].

All arguments and proofs given below can be easily transferred to the case of shells of constant curvatures.

2. Construction of the asymptotic expansion. Let us introduce the following notation. Let the vector $\mathbf{V} \equiv (F, w)$ be the solution, and $\mathbf{P}_1[\mathbf{V}]$ be the left-hand part of the system (1.1). For the solution of (1.1) we construct asymptotic expansions of the form

$$F = \sum_{s=0}^{n+2} \varepsilon^s F_s + \sum_{s=0}^{n+2} \varepsilon^s h_s^\circ + z_n^{(1)}, \qquad w = \sum_{s=0}^{n} \varepsilon^s w_s + \sum_{s=0}^{n} \varepsilon^s g_s^\circ + z_n^{(2)} \tag{2.1}$$

The functions F_s, w_s are obtained by means of the first iteration process [15]. Namely, we set

$$\mathbf{V}_n \equiv (F^n, w^n), \qquad F^n = \sum_{s=0}^{n} \varepsilon^s F_s, \qquad w^n = \sum_{s=0}^{n} \varepsilon^s w_s \tag{2.2}$$

and require that

$$\mathbf{P}_1[\mathbf{V}_n] = O(\varepsilon^{n+1}) \tag{2.3}$$

Setting the coefficients at $\varepsilon^0, \varepsilon^1, \ldots, \varepsilon^n$ in (2.3) equal to zero we obtain the system of equations (1.6) and (1.7) which yields F_0 and w_0, and for the determination of F_s, w_s we obtain the system

$$\Delta^2 F_s + {}^1/_2 \sum_{k+m=s} [w_k, w_m] = 0 \qquad \Delta^2 w_{s-2} - \sum_{k+m=s} [F_k, w_m] = 0 \quad (2.4)$$
$$(w_{-1} = 0, \qquad s = 1, 2, \ldots, n+2)$$

with boundary conditions

$$w_s|_\Gamma = 0, \qquad F_{s,\tau\tau}|_\Gamma = B_s^{(1)}, \qquad F_{s,n\tau}|_\Gamma = B_s^{(2)} \qquad (s = 0, 1, \ldots, n+2) \quad (2.5)$$

Here $B_s^{(i)}$ $(i = 1, 2)$ are so far unknown functions. The functions F^s, w^s $(s = 0, 1, \ldots)$ do not satisfy all the boundary conditions (1.4) and (1.5) and, consequently, the difference $\mathbf{V} - \mathbf{V}_n$ is not small near the boundary Γ. The inconsistencies which arise in the fulfillment of the boundary conditions (1.4) and (1.5) are compensated by functions of the boundary-layer type h_s° and g_s°, which are determined by means of the second iteration process [15].

Namely, we look for the difference $\mathbf{V} - \mathbf{V}_n$ in the form

$$F - F^n = \sum_{m=0}^{n} \varepsilon^m h_m, \qquad w - w^n = \sum_{m=0}^{n} \varepsilon^m g_m \qquad (2.6)$$

In order to determine the functions h_s, g_s, it is necessary to change from the coordinates x, y to the local coordinates. Let us introduce a coordinate system ρ, φ in the vicinity of the boundary Γ; namely, we construct a system of normals, i.e. vectors AR of length $\eta > 0$, drawn from points A of the arc Γ into the domain D, so that the vector **AR** forms a right angle with the tangent to the arc Γ at point A. If η is sufficiently small the normals will not intersect each other. The coordinate ρ of the point N on the normal AR is equal to the distance AN, and φ is the arc length OA, where O is some point of the arc Γ, for which $\varphi = 0$.

We substitute (2.6) into (1.1), bearing in mind (2.3), and in the resulting expression we pass to the local coordinates

$$\sum_{s=0}^{n} \varepsilon^s \Delta^2 h_s + {}^1/_2 \sum_{k+m=s}^{n} \varepsilon^s [w_m, g_k] + {}^1/_2 \sum_{k+m=s}^{n} \varepsilon^s [g_m, g_k] = O(\varepsilon^{n+1})$$

$$\sum_{s=0}^{n} \varepsilon^{s+2} \Delta^2 g_s - \sum_{k+m=s}^{n} \varepsilon^s [F_m, g_k] - \sum_{k+m=s}^{n} \varepsilon^s [w_m, h_k] - \sum_{k+m=s}^{n} \varepsilon^s [g_k, h_m] = O(\varepsilon^{n+1}) \qquad (2.7)$$

Here

$$g_{x_i x_k} = g_{\rho\rho} \rho_{x_i} \rho_{x_k} + g_{\varphi\rho} (\varphi_{x_i} \rho_{x_k} + \varphi_{x_k} \rho_{x_i}) + g_{\varphi\varphi} \varphi_{x_i} \varphi_{x_k} + g_\rho \rho_{x_i x_k} + g_\varphi \varphi_{x_i x_k}$$
$$(i, k = 1, 2, \; x_1 = x, \; x_2 = y)$$

In the new coordinate system the operator Δ^2 will be of the same order, but will have variable coefficients. Expanding these coefficients into Taylor series with respect to ρ in the vicinity of $\rho = 0$ and making the substitution $\rho = \varepsilon t$, we obtain

$$\Delta^2 u = \varepsilon^{-4}\left(\frac{\partial^4 u}{\partial t^4} + \sum_{i=1}^{N} \varepsilon^i R_i u + \varepsilon^{N+1} R_{N+1} u\right)$$

Here $R_i \ (i \leqslant N)$ and R_{N+1} are linear differential operators, whose respective coefficients are of the form

$$\sum_{j\leqslant i} f_j(\varphi)\, t^j, \qquad \sum_{j\leqslant N+1} d_j(\rho,\varphi)\, t^j$$

where $d_i(\rho,\varphi)$ are functions of ρ, φ independent of ε. Furthermore, let

$$F_m = \sum_{l=0}^{N} F_{ml}\rho^l, \qquad w_m = \sum_{l=0} w_{ml}\rho^l \tag{2.8}$$

be the corresponding expansions of the functions F_s and w_s into Taylor series in the vicinity of $\rho = 0$. Now, in (2.7) and (2.8) we set $\rho = \varepsilon t$, we substitute (2.8) into (2.7) and set the coefficients at the same powers of ε equal to zero. For the determination of h_s, g_s we obtain systems of linear differential equations of the fourth order with coefficients depending on φ. The boundary conditions for the functions h_s, g_s when $t = 0$ are determined by the values of the difference $\mathbf{V} - \mathbf{V}_n$ on the boundary Γ for the corresponding powers of ε^s,

$$g_s|_{\rho=0} = 0, \qquad \left.\frac{\partial g_s}{\partial \rho}\right|_{\rho=0} = -\left.\frac{\partial w_s}{\partial n}\right|_{\Gamma} \qquad (s = 0, 1, \ldots) \tag{2.9}$$

and when $t = \infty$, from the condition of existence of the boundary layer, i.e.

$$\left.\frac{\partial g_s}{\partial t}\right|_{t=\infty} = \left.\frac{\partial h_s}{\partial t}\right|_{t=\infty} = h_s|_{t=\infty} = 0 \qquad (s = 0, 1, 2, \ldots) \tag{2.10}$$

Then from (2.7) for h_0 and h_1 we obtain

$$\frac{\partial^4 h_i}{\partial t^4} = 0, \qquad h_i|_{t=\infty} = \left.\frac{\partial h_i}{\partial t}\right|_{t=\infty} = 0 \qquad (i = 0, 1) \tag{2.11}$$

Hence it follows that

$$h_0 = h_1 \equiv 0 \tag{2.12}$$

Now we determine the functions $B_s^{(i)}(\varphi)$, by setting the coefficients at ε^s ($s = 0, 1, \ldots, n + 2$) equal to zero in Expressions

$$\sum_{s=0}^{n+2} \varepsilon^s (B_s^{(1)} + h_{s,\tau\tau})|_{\rho=0} = T(\varphi), \qquad \sum_{s=0}^{n+2} \varepsilon^s (B_s^{(2)} + h_{s,\rho\tau})|_{\rho=0} = S(\varphi)$$

In perticular, from (2.12) we find that $B_0^{(1)} = T(\varphi)$, $B_0^{(2)} = S(\varphi)$. From here actually follows the correctness of choice of the boundary condition for the positive solution of problem (1.6) and (1.7).

Setting the coefficient at ε^{-2} equal to zero in the relation which is obtained from the second expression (2.7) in conjunction with (2.9) to (2.12) leads to Equation

$$\frac{\partial^4 g_0}{\partial t^4} - a\,\frac{\partial^2 g_0}{\partial t^2} = 0 \tag{2.13}$$

$$g_0|_{t=0} = 0, \qquad \left.\frac{\partial g_0}{\partial t}\right|_{t=0} = -\,\varepsilon\,\left.\frac{\partial w_0}{\partial n}\right|_\Gamma, \qquad \left.\frac{\partial g_0}{\partial t}\right|_{t=\infty} = 0 \tag{2.14}$$

Here a is a function depending on the coordinate φ and defined by the relation

$$a = [F_{0,xx}\,\rho_y^2 + F_{0,yy}\,\rho_x^2 - 2F_{0,xy}\rho_x\rho_y]|_\Gamma = [(\rho_x\varphi_y - \rho_y\varphi_x)^2\,F_{\tau\tau}]|_\Gamma = T(\varphi)$$

In the case when $T(\varphi) > 0$, from (2.13) and (2.14) we obtain

$$\frac{\partial g_0}{\partial \rho} = -\,w_{00}\exp\frac{-\sqrt{T(\varphi)}\,\rho}{\varepsilon}, \qquad w_{00} = \left.\frac{\partial w_0}{\partial n}\right|_\Gamma \tag{2.15}$$

i.e. $\partial g_0/\partial\rho$ is a function of the boundary-layer type of order zero [15]. From (2.15) we find

$$g_0(\rho) = \frac{\varepsilon w_{00}}{\sqrt{T(\varphi)}}\left[\exp\frac{-\sqrt{T(\varphi)}\,\rho}{\varepsilon} - 1\right] \tag{2.16}$$

Furthermore, from the first expression (2.7) in exactly the same way we obtain the equation for h_2

$$\frac{\partial^4 h_2}{\partial t^4} = -\,w_{00;\varphi\varphi}\,g_{0\cdot tt} - {}^1\!/_2\,[g_0, g_0]_{(\varphi,\,t)}, \qquad h_2\Big|_{t=\infty} = \left.\frac{\partial h_2}{\partial t}\right|_{t=\infty} = 0 \tag{2.17}$$

where g_0 is determined in (2.16). Integrating we find

$$h_2 = \varepsilon\,[c_1(t,\varphi)\,e^{-2\sqrt{T(\varphi)}\,t} + c_2(t,\varphi)\,e^{-\sqrt{T(\varphi)}\,t}] \tag{2.18}$$

Here $c_1(t,\varphi)$ and $c_2(t,\varphi)$ are polynomials of the second order with respect to t. We determine the functions g_s $(s = 1,\ldots,n)$ from equations which have the form of (2.13) and (2.14), but are nonhomogeneous, and the functions h_s are found by means of term by term integration of expressions of the form $p_s(t,\varphi)\exp(-k(\varphi)t)$, where $k(\varphi) > 0$, and $p_s(t,\varphi)$ is a polynomial in t of order not higher than s. Using the method of mathematical induction in a manner similar to that of [15], it can be easily shown that h_s and g_s are the functions of the boundary-layer type of integgral order. Finally, let us determine the functions h_s° and g_s° $(s = 1, 2, \ldots)$ in Formulas (2.1). To that end we set

$$h_s^\circ = \psi(\rho/\delta)\,h_s, \qquad g_s^\circ = \psi(\rho/\delta)\,g_s \tag{2.19}$$

Here $\psi(\eta)$ is the smoothing-out function (equal to 1 for $\eta \leqslant {}^1\!/_3$ and to 0 for $\eta \geqslant {}^2\!/_3$).

Thus the process of constructing an asymptotic expansion is reduced to the following. We find the positive solution F_0, w_0 of problem (1.6) and (1.7) and from (2.13) and (2.14) we determine g_0. Then from (2.4) we successively find F_s, w_s $(s = 1,2,\ldots)$, and from the nonhomogeneous equations of the form (2.13) and (2.14) we find g_s. (These equations have not been written down because they are very cumbersome)

Let us now construct the asymptotic expansion for the case when

$$T(A) = S(A) = 0,$$

i.e. when the edge of the plate is free of stresses. As is well known, in that case the boundary conditions for the function F can be reduced to the form

$$F(A) = F_n(A) = 0 \qquad (A \in \Gamma) \tag{2.20}$$

Since $T(A) = 0$ Equations (2.13) and (2.14) are incompatible. This means that the boundary layers of integer order with respect to ε are unsuitable for the description of the edge effect phenomena, and therefore terms of higher order with respect to ε must be taken into account. Let us again consider Equation (2.7) and let us write down the following expression in local coordinates in more detail

$$\sum_{m+k=s} [F_m, g_k]_{(x,y)} = \sum_{m+k=s} \{(\rho_x\varphi_y - \rho_y\varphi_x)^2 [F_m, g_k]_{(\rho,\varphi)} + g_{k,\rho\rho} [(\rho_{xx}\rho_y^2 + \rho_{yy}\rho_x^2 - 2\rho_{xy}\rho_x\rho_y) F_{m,\rho} + (\varphi_{xx}\rho_y^2 + \varphi_{yy}\rho_x^2 - 2\varphi_{xy}\rho_x\rho_y) F_{m,\varphi}] + L\}$$

$$\left(F_m = \sum_{l=0}^{N} F_{ml}\rho^l\right) \tag{2.21}$$

Here under L we include terms which contain lower-order derivatives of the function g_k with respect to ρ. Furthermore, we set $\mu = \varepsilon^{2/3}$ and in (2.7) and (2.21) we make a substitution $\rho = \mu t$. Exactly as in the general case, we find that $h_0 = h_1 = 0$ and $B_0^{(i)} = B_1^{(i)} = 0$ $(i = 1, 2)$. Since $B_1^{(i)} = 0$ we deduce from (2.4) and (2.5) that for $s = 1$ $F_1 = w_1 \equiv 0$. Now, in order to determine g_0 we collect the terms at μ^{-1}, and bearing in mind that $F_1 = w_1 = 0$ everywhere in D, and

$$F_{00,\varphi} = F_{00,\varphi\varphi} = F_{00,\varphi\rho} = F_{00,\varphi\varphi\rho} = F_{00,\rho} = 0$$

on the contour Γ we obtain

$$\frac{\partial^4 g_0}{\partial t^4} - a_1 t F_{0,\rho\rho} \frac{\partial^2 g_0}{\partial t^2} = 0 \tag{2.22}$$

Here

$$a_1 = (\rho_{xx}\rho_y^2 + \rho_{yy}\rho_x^2 - 2\rho_{xy}\rho_x\rho_y)\,|\,\Gamma$$

$$g_0\Big|_{t=0} = 0, \qquad \frac{\partial g_0}{\partial t}\Big|_{t=0} = -\mu\frac{\partial w_0}{\partial n}\Big|_{\Gamma}, \qquad \frac{\partial g_0}{\partial t}\Big|_{t=\infty} = 0 \tag{2.23}$$

If $F_{0,\rho\rho}a_1 = l(\varphi) > 0$, then the solution of problem (2.22) and (2.23) can be obtained in the form

$$\frac{\partial g_0}{\partial t_1} = -\frac{6\mu}{\pi} w_{00}(0,\varphi)\int A_1(-t_1)\,dt_1, \qquad w_{00} = \frac{\partial w_0}{\partial n}\Big|_{\Gamma} \tag{2.24}$$

Here [16]

$$A_1(-t_1) = \int_0^\infty \cos(\tau^3 + t_1\tau)\,d\tau, \qquad t_1 = (3l(\varphi))^{1/3}\ (t > 0)$$

The asymptotic representation of $A_1(-t)$ for large values of t has the form

$$A_1(-t) = {}^1/_2 \sqrt{\pi}\,(3t)^{-1/4} \exp\,[-2({}^1/_3 t)^{3/2}]\,(1 + O\,(t^{-3/2})), \qquad t > 0 \quad (2.25)$$

Let us note that in the case of a circular, symmetrically loaded, rigidly built-in plate (the contour is free of stresses) it is easy to show that $I(\varphi) > 0$ [7]. Indeed, in that case $\rho = 1 - r$, $r^2 = x^2 + y^2 \leqslant 1$ and $a_1 = -1$. It remains to show that $F_{00,\rho\rho} < 0$. That follows directly from Formula (4.10) for $T = 0$ and $r = 1$.

We have
$$F_{00,\rho\rho} = -\frac{1}{2}\int_0^1 \frac{\varphi^2(\tau)\,\tau}{v^2}\,d\tau < 0 \qquad [\varphi(1) = 0]$$

It has been established in this Section, that if (1.9) is fulfilled on the contour (which in turn implies the inequality $F_{\tau\tau}(A) > 0$ $(A \in \Gamma)$), then the asymptotic expansions of solution (2.1) can be formally constructed. Below it will be shown that the existence of the positive solution is the sufficient condition for the existence of the membrane solution, for which the asymptotic expansions are valid. At the same time the estimates of errors for $\varepsilon \to 0$ will be established.

3. Justification of the asymptotic expansions. Existence of the membrane solutions. Let us introduce function spaces.

1. The space $L_p(D)$, composed of functions summable with the power $p > 1$ and with the norm

$$\| f \|_{L_p} = \Big(\int_D | f |^p \,dx\,dy\Big)^{1/P} \qquad (3.1)$$

If the vector-function $\mathbf{V} \equiv (F, w)$, is considered, we will assume that $F, w \in L_2(D)$ if each of its components $\mathbf{V} \in L_2(D)$, and we will define the norm of $\mathbf{V}$ by the relation

$$\| \mathbf{V} \|_{L_2}^2 = \| F \|_{L_2}^2 + \| w \|_{L_2}^2 \qquad (3.2)$$

2. The space H of functions f which satisfy the boundary conditions (1.4) and which possess in the domain D the generalized derivatives of the order $l = 4$ which belong to L_2 with the norm

$$\| f \|_H = \Big(\int_D \sum_{k=0}^{4} \sum_{m+n=k} \Big| \frac{\partial^k f}{\partial x^m \partial y^n}\Big|\,dx\,dy\Big)^{1/2} \qquad (3.3)$$

If the vector-function $\mathbf{V} \equiv (F, w)$ is such that its components $F, w \in H$, then we will say that $\mathbf{V} \in E$ and we will associate $\mathbf{V} \in H$ with the norm

$$\| \mathbf{V} \|_H^2 = \| F \|_H^2 + \| w \|_H^2 \qquad (3.4)$$

3. The space $C^{(m)}$ of functions f continuously differentiable m times right up to the contour. The norm in $C^{(m)}$ is defined by Formula

$$\| f \|_{C^{(m)}} = \sum_{\substack{k=1\\ \alpha+\beta=k}}^{m} \max \Big| \frac{\partial^k f}{\partial x^\alpha\,\partial y^\beta}\Big| + \max |f|$$

Let f be an arbitrary, four times continuously differentiable function, which satisfies the conditions (1.5) on Γ . We set

$$F_1 = F - f, \qquad w_1 = w \tag{3.5}$$

Then (1.1) to (1.5) are reduced to the problem

$$\begin{gathered}\triangle^2 F_1 + {}^1/_2\,[w_1, w_1] + \triangle^2 f = 0 \\ \varepsilon^2 \triangle^2 w_1 - [w_1, F_1] - [w_1, f] - q = 0\end{gathered} \tag{3.6}$$

with homogeneous boundary conditions

$$w_1(A) = w_{1,n}(A) = F_1(A) = F_{1,n}(A) = 0 \qquad (A \in \Gamma) \tag{3.7}$$

We look upon problem (3.6) and (3.7) as a functional equation

$$\mathbf{P}\,[\mathbf{V}] = 0 \tag{3.8}$$

Here $\mathbf{V} \equiv (F_1, w_1)$ and has been introduced into (3.5), and the operator $\mathbf{P}$ is defined by the left-hand part of the system with (3.6). It is easy to show that the operator $\mathbf{P}$ acts from the space H upon the space L_2,

For the presentation below it is convenient to introduce the following designations into (2.1)

$$\varphi_k = F - z_k^{(1)}, \qquad \psi_k = w - z_k^{(2)} \tag{3.9}$$

L e m m a 3.1 . The following estimates* are valid for φ_k and ψ_k

$$\triangle^2\varphi_k + {}^1/_2\,[\varphi_k, \psi_k] = O\,(\varepsilon^{k+1}), \qquad \varepsilon^2\triangle^2\psi_k - [\varphi_k, \psi_k] - q = O\,(\varepsilon^{k+1}) \tag{3.10}$$

We will omit the detailed proof and will only note that to obtain the estimates (3.10) one has to substitute the values of φ_k, ψ_k into the left-hand part of (1.1) and make two separate estimates: one in the vicinity of the boundary, in the narrow strip D_δ ($\rho < \delta$) and the other inside the region, i.e. in $D - D_\delta$. The estimate (3.10) for the region $D - D_\delta$ follows directly from (2.3) if one remembers that functions h_i° and g_i°, of the boundary-layer type, are equal to zero outside of the region D_δ . The estimate for the strip D_δ is carried out in exactly the same way as in the case of a circular plate [7].

L e m m a 3.2 . Let $a(x,y)$, $b(x,y)$ and $c(x,y)$ be twice continuously differentiable functions in the region D . Then

$$1) \quad \int_D [a, b]\, c\,dx\,dy = \int_D [a, c]\, b\,dx\,dy \tag{3.11}$$

if $\quad c\,(A) = b\,(A) = 0 \quad (A \in \Gamma)$

$$2) \quad \int_D [a, b]\, a\,dx\,dy = -\int_D (b_{xx}a_y^2 + b_{yy}a_x^2 - 2b_{xy}a_x a_y)\,dx\,dy \tag{3.12}$$

if $\quad a\,(A) = 0 \quad (A \in \Gamma)$.

The proof of the lemma is easily furnished in both cases through integration by parts.

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* The condition $f(\varepsilon) = O(\varepsilon^{k+1})$ means that $|f(\varepsilon)| \leqslant m\varepsilon^{k+1}$.

T h e o r e m 3.1 . The positive solution of problem (1.6) and (1.7) is unique.

P r o o f . We will take the equations of a membrane in the form (3.6) and (3.7), setting $\varepsilon = 0$ in (3.6) and neglecting the boundary condition for $w_{1,n}$ in (3.7).

Let us assume the existence of two solutions $\mathbf{V}^{(1)}$ and $\mathbf{V}^{(2)}$. Then we have $\mathbf{P}[\mathbf{V}^{(1)}]=0$ and $\mathbf{P}[\mathbf{V}^{(2)}]=0$. Now we subtract one equality from the other and multiply by the difference $\mathbf{V}^{(1)}-\mathbf{V}^{(2)}$, then we integrate over the region D and add. Applying Lemma 3.2 we obtain

$$\int_D [\Delta(F_1^{(1)}-F_1^{(2)})]^2\,dx\,dy + \int_D [(F_1^{(1)}+F_1^{(2)})_{xx}(w_1^{(1)}-w_1^{(2)})_y^2 + (F_1^{(1)}+F_1^{(2)})_{yy} \times$$
$$\times (w_1^{(1)}-w_1^{(2)})_x^2 - 2(F_1^{(1)}+F_1^{(2)})_{xy}(w_1^{(1)}-w_1^{(2)})_x(w_1^{(1)}-w_1^{(2)})_y]\,dy\,dx = 0 \quad (3.13)$$

Bearing in mind that (1.9) is valid for both solutions $F_1^{(1)}$ and $F_1^{(2)}$, we arrive at the conclusion that the second integral in (3.13)is non-negative; hence $F_1^{(1)} \equiv F_1^{(2)}$.

T h e o r e m 3.2 . If problem (1.6) and (1.7) has the positive solution, then problem (1.1) to (1.5) has one and only one membrane solution.

P r o o f . The uniqueness is proved in exactly the same way as the uniqueness of the positive solution for a membrane. To prove the existence of the membrane solution one applies a theorem due to Kantorovich [12] on the convergence of the Newton's method for the operator equations. The first approximation is taken as $\mathbf{V}_k^* \equiv (\varphi_k - f, \psi_k)$. As applied to this problem the theorem has the following form.

T h e o r e m . Let the operator $\mathbf{P}$ be defined in a sphere

$$\Omega\,(\|\mathbf{V}-\mathbf{V}_k^*\| \leqslant R)$$

of the space H and have a continuous second derivative in the closed sphere $\Omega_0\,(\|\mathbf{V}-\mathbf{V}_k^*\| \leqslant r)$ Moreover, let

1) the linear operation $\Gamma_0 = [\mathbf{P}_{\mathbf{V}_k^*}(\mathbf{V})]^{-1}$ exist

2) $\Gamma_0(\mathbf{P}[\mathbf{V}_k^*])\|_H \leqslant \eta$

3) $\|\Gamma_0(\mathbf{P}''(\mathbf{V}))\|_H \leqslant K$

4) $h = K\eta \leqslant 1/2,\ r \geqslant r_0 = (1-\sqrt{1-2h})\,h^{-1}\eta$

Then Equation (3.8) has the solution $\mathbf{V}^*$ to which the process converges. Here

$$\|\mathbf{V}-\mathbf{V}_k^*\|_H \leqslant r_0 \quad (3.14)$$

The conditions of the theorem are obviously satisfied if

$$\|\mathbf{P}(\mathbf{V}_k^*)\|_{L_2}\|(\mathbf{P}'_{\mathbf{V}_k^*})^{-1}\|^2\|\mathbf{P}_{\mathbf{V}}''\| \leqslant 1/2 \quad (3.15)$$

Let us show that (3.15) is fulfilled for sufficiently small ε for any $k > 5$. From Lemma 3.1 we derive *

$$\|\mathbf{P}(\mathbf{V}_k^*)\| \leqslant m_1\varepsilon^{k+1} \quad (3.16)$$

* Here and everywhere henceforth m_i are certain constants, independent of ε.

In order to estimate the second factor in (3.15), let us consider the linear equation

$$\mathbf{P}_{\mathbf{V}_k}{}'{}^*(\delta\mathbf{V}) = \mathbf{f}, \qquad \delta\mathbf{V} \equiv (\delta F, \delta w), \qquad \mathbf{f} \equiv (f_1, f_2) \tag{3.17}$$

$$\mathbf{P}_{\mathbf{V}_k}{}'{}^*(\delta\mathbf{V}) \equiv (\Delta^2(\delta F) + [\psi_k, \delta w],\ \varepsilon^2\Delta^2(\delta w) - [\psi_k, \delta F] - [\varphi_k, \delta w])$$

$$\delta F\,|_\Gamma = (\delta F)_n\,|_\Gamma = \delta w\,|_\Gamma = (\delta w)_n\,|_\Gamma = 0 \tag{3.18}$$

$$\mathbf{f} \in L_2, \qquad \mathbf{P}_{\mathbf{V}_k}{}'{}^* \in [H \to L_2]$$

From (3.17) and (3.18) we obtain

$$\int\limits_D (\triangle(\delta F))^2\,dx\,dy + \varepsilon^2 \int\limits_D (\triangle(\delta w))^2\,dx\,dy + \int\limits_D [\varphi_{k,\,xx}(\delta w_y)^2 + \varphi_{k,\,yy}(\delta w_x)^2 -$$

$$- 2\varphi_{k,\,xy}\delta w_x \delta w_y]\,dx\,dy = \int\limits_D (f_1\delta F + f_2\delta w)\,dx\,dy \tag{3.19}$$

Let us now prove that for sufficiently small ε the last integral in (3.17) is positive. To show that we look at the second derivatives $\varphi_{k,\,xx}$, $\varphi_{k,\,yy}$, $\varphi_{k,\,xy}$. For instance, for $\varphi_{k,\,xx}$ we have

$$\varphi_{k,\,xx} = F_{0,\,xx} + \sum_{s=1}^{k+2} \varepsilon^s F_{s,\,xx} + \sum_{s=0}^{k+2} \varepsilon^s h_{s,\,xx}$$

Bearing in mind that $h_0 = h_1 = 0$ and $\varepsilon^2 h_{2,\,xx} = O(\varepsilon)$ (see (2.18)), we easily obtain

$$\varphi_{k,\,xx} = F_{0,\,xx} + O(\varepsilon) \tag{3.20}$$

In the same manner we prove the validity of the relations

$$\varphi_{k,\,yy} = F_{0,\,yy} + O(\varepsilon), \qquad \varphi_{k,\,xy} = F_{0,\,xy} + O(\varepsilon) \tag{3.21}$$

Now let us note that from (1.9) follows directly the estimate

$$\int\limits_D (F_{0,\,xx}u_y^2 + F_{0,\,yy}u_x^2 - 2F_{0,\,xy}u_x u_y)\,dx\,dy \geqslant m_2 \int\limits_D |\nabla u|^2 dx\,dy \qquad (m_2 > 0) \tag{3.22}$$

Finally, by means of (3.20) to (3.22) for sufficiently small ε (3.23)

$$\int\limits_D [\varphi_{k,\,xx}(\delta w_y)^2 + \varphi_{k,\,yy}(\delta w_x)^2 - 2\varphi_{k,\,xy}\delta w_x \delta w_y]\,dx\,dy \geqslant m_3 \int\limits_D |\nabla\delta w|^2\,dx\,dy, \qquad m_3 > 0$$

Now, from (3.19), using (3.23), we derive

$$\|\triangle(\delta F)\|_{L_2} \leqslant m_4 \|\mathbf{f}\|_{L_2}, \qquad \|\triangle(\delta w)\|_{L_2} \leqslant m_4 / \varepsilon\, \|f\|_{L_2} \tag{3.24}$$

Let us also note the following inequalities for φ_k and ψ_k which easily follow from (3.20), (3.21) and (2.16)

$$\|\varphi_k\|_{C(2)} \leqslant m_5, \qquad \|\psi_k\|_{C(2)} \leqslant m_6 / \varepsilon \tag{3.25}$$

Furthermore, from (3.17) we obtain the estimates

$$\|\triangle^2(\delta F)\|_{L_2}{}^2 \leqslant 2\,(\|f_1\|_{L_2}{}^2 + \|[\delta w,\ \psi_k]\|_{L_2}{}^2) \tag{3.26}$$

$$\|\triangle^2(\delta w)\|_{L_2}{}^2 \leqslant 3/\varepsilon^4\,(\|f_2\|_{L_2}{}^2 + \|[\delta w,\ \varphi_k]\|_{L_2}{}^2 + \|[\delta F,\ \psi_k]\|_{L_2}{}^2) \tag{3.27}$$

Hence, using the results from [17] and the estimates (3.24), (3.25) we find

$$\|\delta F\|_H \leqslant m_7 / \varepsilon^2\, \|\mathbf{f}\|_{L_2}, \qquad \|\delta w\|_H \leqslant m_8 / \varepsilon^3\, \|\mathbf{f}\|_{L_2} \tag{3.28}$$

From (3.17) and (3.28) it is easy to show that the operator $P'_{V_{k^*}}$ is invertible and that the following estimate is valid

$$\| (\mathbf{P}_{\mathbf{V}_{k^*}}')^{-1} \| \leqslant m_9 / \varepsilon^3 \tag{3.29}$$

To form an estimate of $\| P_{\mathbf{V}}'' \|$ we consider the bilinear form

$$\mathbf{P}_{\mathbf{V}}'' (\delta\mathbf{V}) (\delta_1\mathbf{V})^3 = ([\delta w, \delta_1 w], - [\delta w, \delta_1 F] - [\delta_1, w, \delta F])$$

Using the insertion theorems [18] we derive

$$\| \mathbf{P}_{\mathbf{V}}'' (\delta\mathbf{V}) (\delta_1\mathbf{V}) \|_{L_2}^2 \leqslant m_{10} \| \delta V \|_H^2 \| \delta_1 V \|_H^2$$

and hence follows the estimate

$$\| \mathbf{P}_{\mathbf{V}}'' \| \leqslant m_{11} \tag{3.30}$$

From (3.16), (3.29) and (3.30) we obtain

$$\| \mathbf{P} (\mathbf{V}_k^*) \|_{L_2} \| (\mathbf{P}_{\mathbf{V}_{k^*}}')^{-1} \|^2 \| \mathbf{P}_{\mathbf{V}}'' \| \leqslant m_{12} \varepsilon^{k-5} < 1/2 \tag{3.31}$$

provided that $k > 5$ and ε is sufficiently small $(0 < \varepsilon < \varepsilon_1)$.

And so the conditions of the Kantorovich theorem are fulfilled. Therefore, Equation (3.8), equivalent to the problem (1.1) to (1.6), has the solution $\mathbf{V}^* \equiv (F_1^*, w_1^*)$ for which the estimates (3.14) are valid. We compute the quantity r_0 by means of (3.16) and (3.29)

$$\| \mathbf{V}^* - \mathbf{V}_k^* \| \leqslant m_{13} \varepsilon^{k-2} \qquad (k > 5) \tag{3.32}$$

Due to the insertion theorems [18] we have from (3.32)

$$\| z_k^{(i)} \|_{C(l)} < m_{14} \varepsilon^{k-2} \qquad (k > 5, \ i = 1, 2, \ l = 0, 1, 2) \tag{3.33}$$

Finally, from (3.33) for $l = 2$, $i = 1$, using (3.20), (3.21) we obtain

$$F_{xx} = F_{0,\, xx} + O(\varepsilon), \qquad F_{yy} = F_{0,\, yy} + O(\varepsilon), \quad F_{xy} = F_{0,\, xy} + O(\varepsilon) \tag{3.34}$$

It follows from here that the inequality (1.9) is valid. This means that the solution $\mathbf{V}^*$ constructed above is a membrane solution. Theorem 3.2 is thus proved. Along with the proof of Theorem 3.2 the following corollaries have been arrived at.

T h e o r e m 3.3 . For the membrane solution of problem (1.1) to (1.6) the valid asymptotic representations are given by (2.1) and the remainders allow the following estimates

$$\| z_k^{(1)} \|_{C(l)} \leqslant m_{15} \varepsilon^{k+1} \qquad (k = 0, 1, 2, \ldots, \ l = 0, 1, 2) \tag{3.35}$$

$$\| z_k^{(2)} \|_{C(1)} \leqslant m_{16} \varepsilon^{k+1} \qquad (k = 0, 1, \ldots) \tag{3.36}$$

$$\| z_k^{(2)} \|_{C(2)} \leqslant m_{17} \varepsilon^{k} \qquad (k = 1, 2, \ldots) \tag{3.37}$$

The estimates (3.35) to (3.37) follow directly from (3.33) by means of the triangle inequality, and bearing in mind that each differentiation of functions h_s°, g_s° of the boundary-layer type lowers their order in ε by one.

4. On the membrane equations. The existence theorems for the positive solutions of problem (1.6),(1.7) will be obtained below for certain cases.

1. Consider a circular-symmetrically loaded membrane. Let the x-axis coincide with the direction of the radius vector for $\varphi = 0$. Then, taking advantage of the radial symmetry and eliminating the function $w_{0,r}$ from (1.6) we obtain Equation

$$- r \frac{d}{dr} \frac{1}{r} \frac{d}{dr} rv = \frac{\varphi^2}{2v^2}, \qquad \frac{v}{r}\Big|_{r=0} < \infty, \qquad \varphi(r) = \int_0^r q(t)\, t\, dt \tag{4.1}$$

Here $v = F_{0,r}$ is the radial force. The boundary conditions are determined by the manner in which the membrane is supported along the contour. We designate the stresses in the middle surface acting along the radius and along the arc by σ_r and σ_φ, respectively. The latter are expressed in terms of the stress function F_0 by Formulas

$$\sigma = \frac{1}{r} F_{0,r}, \qquad \sigma_\varphi = F_{0,rr} \qquad (\tau = 0) \tag{4.2}$$

Here (1.9) becomes the inequality

$$\sigma_r \sigma_\varphi > 0 \tag{4.3}$$

a) Let the membrane be rigidly built-in along the contour

$$dv \mathbin{/} dr - (\nu \mathbin{/} r)_v \big|_{r=1} = 0 \qquad (0 < \nu < 0.5) \tag{4.4}$$

T h e o r e m 4.1 . The problem (4.1) and (4.4) has the positive solution.

P r o o f . The existence of the solution of problem (4.1), (4.4) has been proved in [7]. It remains to prove (4.3). To that end we transfer from (4.1), (4.4) to an equivalent integral equation

$$r\sigma_r = v = \frac{1}{2} r^{-1} J(r, 1) + r \frac{1+\nu}{2(1-\nu)} J(1, 1) \tag{4.5}$$

The following designation has been introduced here

$$J(r, s) = \int_0^r \eta \int_\eta^s \frac{\varphi^2}{\xi v^2} d\xi \, d\eta$$

Differentiating (4.5) with respect to r we obtain

$$\sigma_\varphi = -\frac{1}{2} r^{-2} J(r, 1) + \frac{1}{2} \int_r^1 \frac{\varphi^2}{\xi v^2} d\xi + \frac{1+\nu}{2(1-\nu)} J(1,1) \tag{4.6}$$

In the interval [0,1] Expression $\Phi(r) = -{}^1/_2\, r^{-2} J(r, 1)$ is a decreasing function in r , since $\Phi(0) = 0$ and

$$\partial\Phi \mathbin{/} \partial r = r^{-3} J(r, r)$$

Therefore, the minimum value of $\Phi(r)$ is attained at point $r = 1$. Furthermore, obviously

$$|\Phi(r)| \leqslant |\Phi(1)| \leqslant \frac{1+\nu}{2(1-\nu)} J(1, 1) \qquad (0 < \nu < 0.5) \tag{4.7}$$

Finally, from (4.6) with the help of (4.7) we find that $\sigma_\varphi > 0$, if $r \in (0, 1]$. It is evident that σ_r is positive and the condition (4.3) is fulfilled.

C o r o l l a r y 1 . The symmetrical solution of the problem of large deflections of a circular-symmetrically loaded plate, rigidly built-in along the contour is a membrane solution. (This follows from Theorem 3.2). It has been proved in [10] that in the case of a circular-symmetrically loaded plate, rigidly built-in along the contour, for sufficiently large $q(r)$ a nonsymmetrical solution appears along with the symmetrical one. From the corollary 1 and Theorem 3.2 follows.

C o r o l l a r y 2 . The nonsymmetrical solution will not be a membrane solution.

b) Let the membrane be subjected to a uniform normal tension on the contour

$$v \big|_{r=1} = T \geqslant 0 \tag{4.8}$$

In the case when $T = 0$, the contour is free of stresses.

Theorem 4.2. Let the following inequality be fulfilled

$$T > T_0 = \max_{0 \leqslant r \leqslant 1} \left(\frac{1}{2r^2} \Phi(r) \right)^{1/3} \qquad \left(\Phi(r) = \int_0^r \frac{\varphi^2(\tau)}{\tau} d\tau \right) \tag{4.9}$$

Then problem (4.1), (4.8) has a unique positive solution.

Proof. The existence and uniqueness of solution of problem (4.1), (4.8) have been proved in [7]. Let us find when condition (4.3) is valid. For that we transfer to an equivalent integral equation

$$r\sigma_r = v \quad \frac{r}{2} \int_r^1 \Psi(t) \frac{dt}{t^3} + Tr \qquad \left(\Psi(t) = \int_0^t \frac{\tau\varphi^2(\tau)}{v^2} d\tau \right) \tag{4.10}$$

Differentiating (4.9) with respect to r, we obtain

$$\sigma_\varphi = \frac{1}{2} \int_r^1 \Psi(t) \frac{dt}{t^3} - \frac{1}{2r^2} \Psi(r) + T \tag{4.11}$$

Let $T > 0$. Then from (4.10) we find that $v \geqslant Tr$. Utilizing this we find from (4.11) that if $\frac{1}{2} \Phi(r) \leqslant T^3 r^2$, then σ_φ is positive. Therefore, when $T > T_0$, condition (4.3) is fulfilled and the solution is positive.

Corollary 1. If $0 \leqslant T < T_0$, the solution of problem (4.1), (4.8) will not be positive.

Corollary 2. If $T > T_0$, then the symmetrical solution for a circular symmetrically-loaded plate subjected to the tension T on the contour will be a membrane solution. If, however, $0 \leqslant T < T_0$, the symmertical solution will not be a membrane one.

But if we restrict ourselves to the set of functions which depend on r only and if we consider a solution which satisfies the condition $\sigma_r \geqslant 0$, to be the membrane solution, then it can be shown that the symmetrical solution will be the membrane one for any $T \geqslant 0$ [7]. The results given in [7] and other, somewhat more exact ones, can be obtained by means of Theorems 3.2 and 3.3 which were proved for the one-dimensional case in the investigation of the asymptotic expansions of the solutions of symmetrically loaded shells of revolution [13]. In this connection it is important to point out the following fact. The justification of validity of the asymptotic expansion is not related to the method by which it was constructed. The fulfillment of the estimates (3.16), (3.29) and (3.15) turns out to be the only essential matter.

2. Let the following stresses be given on the boundary of a membrane of arbitrary shape

$$F_{0,\tau\tau}|_\Gamma = a\cos^2\theta + b\sin^2\theta, \qquad F_{0,n\tau}|_\Gamma = \tfrac{1}{2}(b-a)\sin 2\theta \qquad (a > 0, \quad b > 0) \tag{4.12}$$

Here $\theta = \theta(A)$ is the angle which the normal n forms with the x-axis. In the case when $a = b = \sigma$, Problem (1.1) to (1.4) and (4.12) becomes problem (1.1) to (1.4) and (1.8), which was investigated by Fife in [8].

Let us transform the initial equations. We set

$$F_1 = F_0\sigma^{-1}, \qquad F = F_1 - \tfrac{1}{2}(a_1x^2 + b_1y^2) \tag{4.13}$$

Here

$$a_1 = a\sigma^{-1/2} > 0, \quad b_1 = b\sigma^{-1/2} > 0, \quad \sigma = 1/2\,(a^2 + b^2)$$
$$w = w_0\sigma^{-1/2}, \quad p = q\sigma^{-3/2}, \quad \varepsilon_1 = \varepsilon\sigma^{-1/2} \tag{4.14}$$

Then (1.6) and (4.12) becomes problem

$$\Delta^2 F + 1/2\,[w, w] = 0, \quad -a_1 w_{xx} - b_1 w_{yy} - [F, w] - p = 0 \tag{4.15}$$

$$w\,(A) = F\,(A) = F_n\,(A) = 0, \quad (A \in \Gamma) \tag{4.16}$$

Repeating the argument of [8] it is easy to prove the existence of solutions of problem (4.15), (4.16) for sufficiently small $p(x,y)$. The proof is obtained by utilizing the spaces $B^{l,\alpha}$ of functions f, determined in D with continuous derivatives of order l, which uniformly satisfy Hölder's condition with the exponent $\alpha(0 < \alpha < 1)$ along the entire contour with the norm

$$\| f \|_{l+\alpha} = \| f \|_{\bar{B}^{l,\alpha}} = \sup |D^l f| + \sup \frac{|D^l f(N_1) - D^l f(N_2)|}{|N_1 - N_2|^\alpha} \tag{4.17}$$

It is assumed here that $N_1 = (\varphi_1, n)$, $N_2 = (\varphi_2, n)$ and the upper bound is taken over all points $N_1 \neq N_2$ from D and over all derivatives of order l.

T h e o r e m 4.3 . Let $\|p\|_{l+\alpha} < p_0$. Then, if p_0 is sufficiently small, there exists a solution of problem (4.15), (4.16), such that

$$\| F \|_{l+4+\alpha} + \| w \|_{l+2+\alpha} \leqslant \| p \|_{l+\alpha}^{1/2} \tag{4.18}$$

P r o o f * . Let us define the sequence F^i, w^i by formulas

$$F^\circ = w^\circ = 0$$

$$\Delta^2 F^i = -1/2\,[w^{i-1}, w^{i-1}] \quad (i = 1, 2, \ldots) \tag{4.19}$$

$$-a_1 w_{xx}^i - b_1 w_{yy}^i = p\,(x, y) + [F^{i-1}, w^{i-1}] \tag{4.20}$$

$$w^i(A) = F^i(A) = F_n^i(A) = 0 \tag{4.21}$$

Furthermore, we introduce the designations

$$A_i = \| F^i \|_{l+4+\alpha} + \| w^i \|_{l+2+\alpha} \tag{4.22}$$

By means of arguments of [8] and also the theorem 7.3 of [19], it can be shown that

$$A_i < C_1\,(A_{i-1}^2 + \| p \|_{l+\alpha}) \tag{4.23}$$

Then we choose the constant C_1 such that

$$p_0 < 1/4 C_1^2 \tag{4.24}$$

Now, it follows from (4.23) and (4.24) that

$$A_i \leqslant \| p \|_{l+\alpha}^{1/2} \quad (i = 1, 2, \ldots) \tag{4.25}$$

In order to show that the sequences F^i, w^i converge to the solution, we determine the differences

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* For $a_1 = b_1 = 1$ Theorem 4.3 was proved in [8] (Theorem 2).

$$\delta^i F = F^i - F^{i-1}, \qquad \delta^i w = w^i - w^{i-1} \tag{4.26}$$

Now we replace the index i by $i-1$ in Equations (4.19) to (4.21) and subtract the resulting equations from the original ones. Using Theorem 7.3 from [19] and (4.25) we obtain

$$\| \delta^i F \|_{l+4+\alpha} + \| \delta^i w \|_{l+2+\alpha} < C_2 \| p \|_{l+\alpha}^{1/2} (\| \delta^{i-1} F \|_{l+4+\alpha} + \| \delta^{i-1} w \|_{l+2+\alpha}) \tag{4.27}$$

It follows from (4.27) that in order for the sequence to converge to the solution within the corresponding norms, the following condition must be fulfilled

$$C_2 \sqrt{p_0} < 1 \tag{4.28}$$

From this and from (4.24) it is clear that if p_0 is sufficiently small the solution of the problem (4.15), (4.16) exists and (4.18) is fulfilled.

The following theorem results from the estimate (4.18).

Theorem 4.4. If $\sqrt{p_0} < d = \min(1/_2 a_1, 1/_2 b_1)$, then the solution of problem (1.6), (4.12) will be positive.

Proof. It is easy to derive the following inequalities from (4.18)

$$| F_{xx} | < d, \qquad | F_{yy} | < d, \qquad | F_{xy} | < d \tag{4 29}$$

Hence, bearing (4.13) in mind, we obtain for F_0

$$F_{0,xx} > 1/_2 a \sigma^{1/2}, \qquad F_{0,yy} > 1/_2 b \sigma^{1/2}, \qquad F_{0,xx} F_{0,yy} - F_{0,xy}^2 > 0 \tag{4.30}$$

In the case when $a = b = \sigma$, we have to set $p_0 = \frac{1}{4}$.

Corollary. If $\| p \|_{l+\alpha}^{1/2} < d$, then the solution of problem (1.1) to (1.4) and (4.12) is the membrane solution.

3. As an example let us consider a circular membrane subjected on the contour to the nonsymmetrical tensile stresses of the form

$$\sigma_r |_{r=1} = a + b \sin^2 \varphi, \qquad \tau |_{r=1} = 0 \tag{4.31}$$

Here a and b are constants which satisfy the condition $a > 0$ if

$b > 0$ and $a > |b| > 0$, if $b < 0$.

Equations (1.6) and (4.31) can be reduced to the form

$$\Delta^2 F + 1/_2 [w, w] = 0, \qquad -Lw - [F, w] - q = 0 \tag{4.32}$$

$$w |_{r=1} = F |_{r=1} = \partial F / \partial r |_{r=1} = 0 \tag{4.33}$$

where L is the elliptic operator. This can be done by setting

$$F = F_0 - 1/_2 a r^2 - 1/_4 r^2 b (1 + \cos 2\varphi - 1/_3 r^2 \cos 2\varphi), \qquad w = w_0$$

Now, applying the arguments of Theorems 4.3 and 4.4 to (4.32) and (4.33) we establish the following Theorem.

Theorem 4.5. There exists a constant p_0, such that if $\| q \|_{l+\alpha} < p_0$, the solution of problem (1.6), (4.31) is positive.

In the general case of the problem (1.6), (1.7) the solution is not positive. As an example consider a circular membrane subjected at the ends of a diameter to two radial concentrated loads P, whereas the transverse loads are absent. Then (1.6) transfers into Equation

$$\Delta^2 F_0 = 0$$

The formulas for σ_r, σ_φ and τ, as well as the graphs of these functions plotted versus r and φ, are given in [21](p.612) for this particular case. It is clear from the graphs that σ_r and σ_φ change their signs and hence do not possess the property of being positive. The statement of the corollary 1 of Theorem 4.2 may serve as another example.

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